**Marine Hydrodynamics**

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**Lecture - 27**
**Wave Motion in Two Layer Fluids**

Good afternoon, in the last couple of classes, we have talked about wave motion in a homogenous having in a homogenous fluid and particularly in case of a single layer fluid. The density was assumed, it was fluid is assumed to be incompressible and so we have considered that density has homogenous density. Now, question comes you, if the density is not homogenous, particularly in the real ocean the density here is and that varies due to stratification (( )) due to change in temperature.

That can be due to because in the other theories, we have river water enter to the sea and then we have a because of fresh water enters mix with the saline water. So, mixing takes place and in the process, there are many places, where the there is a fresh layer of there is layer of fresh water on the top of the saline water. And also there is a kind of circulation in the ocean, so all these things leads to a stratified fluid.

However, today we will discuss, as if we have seen that as if there are two layer of fluid, the top layer is a top layer is a lower density then the bottom layer and this concept in fact, is used in the transportation of oil and various other chemicals and chemicals also. So, to do these, today only emphasis we will start with, as if we consider a wave it consider the wave motion in a two layer fluid.

Initially, we will start with, as if there is a there are the two layer of the fluids are covered by a common covered by leads. And then only on the top layer, if you will say that, it is exposed to the atmosphere whereas, the bottom is a rigid body or a or a rigid bottom then, what will happen. Particularly, when we have we have seen that, when we have a single layer fluid having a free surface, we have seen there is a one wave, whether it is a case of long wave or it is the case of short waves.

We have analyzed both the case of small amplitude waves, we initially started with the waves of general nature, but of small amplitude. But, there we have seen in the last couple of classes about long wave linearized long wave. So, today same thing we will see, what happened in case of, when there are two fluids of different density having a common interface, let us have a look at it how it goes.

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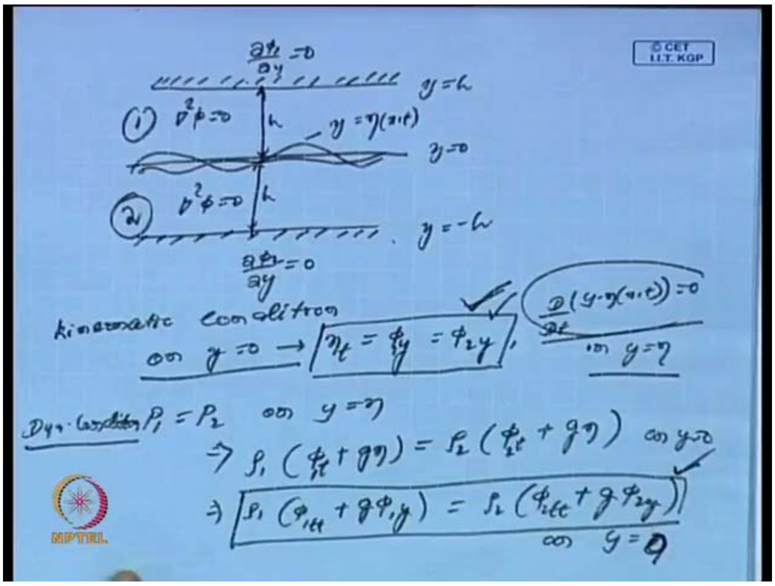


As I have told, I will start with a case of a two layer fluid, with which are covered by two lids, which are all the rigid boundary so, I call them as a lid. And then, let me say suppose, y is equal to h is one of the for sake of simplicity, I will take both of the layers of same height sorry so, this is y is equal to 0, this is y is equal to minus h. Suppose, I assume the heights are same, this is height h, this height h also h. So, as usual, when we have a interface, here in both the surface because, we are asking in mixed fluid so, del square phi is 0.

And here also, we are assuming here also del square phi is 0 in both the layers, let me assume that the interface is y is equal to eta x t, this is the interface. Then, this is a rigid lid, this is a rigid lid so, here del phi by del y will be 0 and again here, del phi by del y is… Hence, we have seen that, if y is equal to eta, as usual on a interface or a surface, we have two types of boundary condition, one is the kinematic condition and the other is the dynamic condition.

The kinematic condition gives us eta t is equal to phi y and if I say this upper layer as layer 1 and the lower layer is a layer 2 then, I call this, I denote the velocity potential in the upper layer by eta phi 1 and here as phi 2. The second layer as phi 2 so then, I can this as eta t is a phi 1 y is a phi 2 y because, eta t is a phi y on the interface and this comes from the kinematic condition. That means, I assuming D by D t y minus eta is 0, assuming there is no gap between the two layers.

So, this this you linearize, this is the satisfied on y is equal to eta and if you linearize this then, we get on y is equal to 0. We have this condition, eta t is phi y is one of the condition, it is the kinematic condition. And again, on that dynamic condition, if you look at the linearized dynamic condition, that comes from the Bernoulli's equation that means, if I say at the interface, we have P 1 is equal to P 2.

P 1 is the, P is the hydrodynamic pressure and then, that will give me in the P 1 is P 2 on y is equal to eta then, if I apply the Bernoulli's equation, that will give me rho 1 into phi t plus g eta phi t plus g eta is equal to rho 2 phi t plus g eta and this rho phi 2, this we call phi 2, this is called phi 1. And again, if I utilize, I take the derivative with respect to t and utilize the kinematic condition, this comes from the dynamic condition.

This is my dynamic condition hence, this will give me, what it will give me so, if I take the derivative with t, that will give me rho 1 phi 1 t t plus g eta t, eta t is phi y so, I will get rho 1 phi 1 t plus g phi y is equal to rho 2 phi 2 t t plus g phi 2 y. So, and this condition will be again on y is equal to eta and since sorry this is the y is equal to eta and this condition is the linearized condition, this is becoming on y is equal to 0 because, after linearization and using the Taylor series, you can get this on y is equal to 0.

So, this is on y is equal to eta, which also can be, this is the linearized condition so, P 1 is P 2, so this will on y is equal to 0 because, in linearization, we will get eta and then, this will be one, y is equal to 0. So, this is the condition 2, conditions the linearized boundary conditions one is the kinematic condition, this is the kinematic condition and this is the dynamic condition are the interface of the fluid. And if that happens then, what does it gives me and again I have told, in case of finite water depth because, this is covered by two lids, here this is del phi 1 by del y is 0 and here, it is del phi 2 by del y 0.

So, if I apply these two conditions then, the other two conditions, which is satisfied at the interface of the fluid so, I will summarize this. I have a lid, there are two fluids, density rho 1 and rho 2 and then, that two lids, the bottom lid is at phi is equal to minus h and the top lid is at y is equal to h. Here, del phi 1 by del y is 0 because, the fluid is not pass through this and your del phi 2 by del y 0. At the interface we have two condition, one is the kinematic condition and the linearized kinematic condition becomes eta t is a phi y’s, which is same as eta t is phi 1 by phi 2 y.

And that comes from the condition that, there is a long gap between at the interface of the two fluids that is, on y is equal near y is equal to eta. Again, we have the dynamic condition P 1 is P 2 on y is equal to eta and that gives us this condition because here, this from the Bernoulli equation, we get this. And again, we utilize the kinematic condition, which gives us this so, the question is that, if I just summarize the whole thing as a boundary value problem, if I summarize the whole thing as a boundary value problem.

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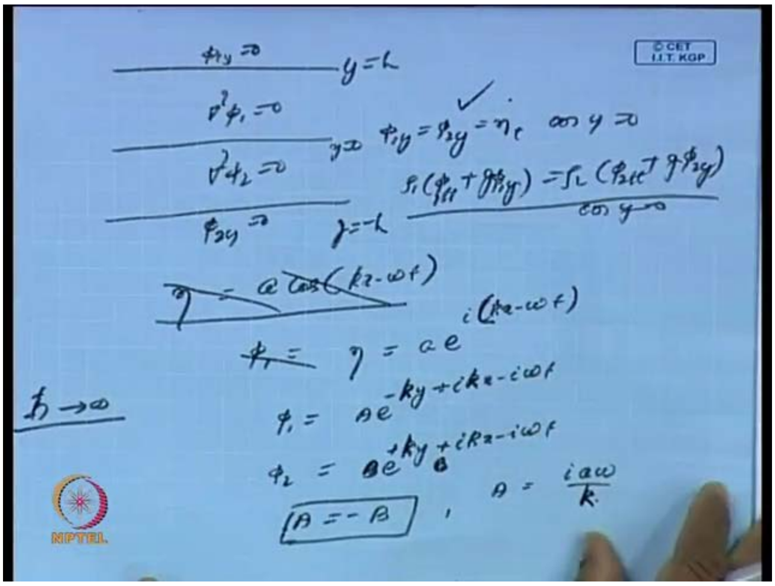


So, I have this, I have phi 1 y is 0 and del square phi is 0 and on the interface, I have phi 1 y is phi 2 y is eta t on y is equal to 0. And another condition is, rho 1 phi 1 t t plus g phi 1 y is rho 2 phi 2 t t plus g phi 2 y and this is what, this is both the condition is satisfied on y is equal to 0. And here also, del square phi 2 is 0 and here we have phi 2 y is 0, this is y is equal to h, this is this line is y is equal to 0 and this is y is equal to minus h. So now, it becomes two radium having a common interface and now suppose, I start with eta is equal to, at the interface suppose I say, whether a wave form exists or not.

If I assume that, eta is equal to a cos k x minus omega t then, what will happen to my phi 1 because, if I have eta, the corresponding phi from the theory of water waves I know on my first few classes, I know my phi 1 can be of this problem or a let me rather take the most general form. I will here concentrate on two cases sorry, I consider the most general form which two results can come together, e to the i times k x minus omega t.

Hence, more simpler to analyze and I say that, initial l case consider that, if h is infinity, the depth is infinity that means, water depth is very large if if you will go for the interface whether on the top layer or the bottom layer. So, in that case what will happen, I can have phi 1 of this one, a e to the minus k y plus i k x minus i omega t. And my phi 2 suppose, I say then, it will be a e to the minus k y plus k y e to the power plus i k x minus i omega t.

So, because of this condition you can say that, this will be and since this if you look at this, this satisfy the Laplace equation in both the layers and also if you look at eta, as you have seen that, if eta is of this form then, phi will be of this form. Now, if you look at the kinematic equation phi 1 minus phi 2 y, that will give us a is equal to minus b. And another thing is that, once a is equal to minus b now, if I look at the again it can be easily established that, if i phi 1 is eta t if I utilize this then, I will get a small relation between a and omega, a is equal to i a omega by k, call it k y minus omega t sorry this is k that small k.

So, a is equal to i omega by k, it can be easily seen if you again utilize this condition so, we have two relation we have got. Now, another thing if I apply this, if I substitute for phi 1 phi 2 in the second equation, what I will get particularly, in this equation on y is equal to 0, I will get one more relation, that relation gives me.

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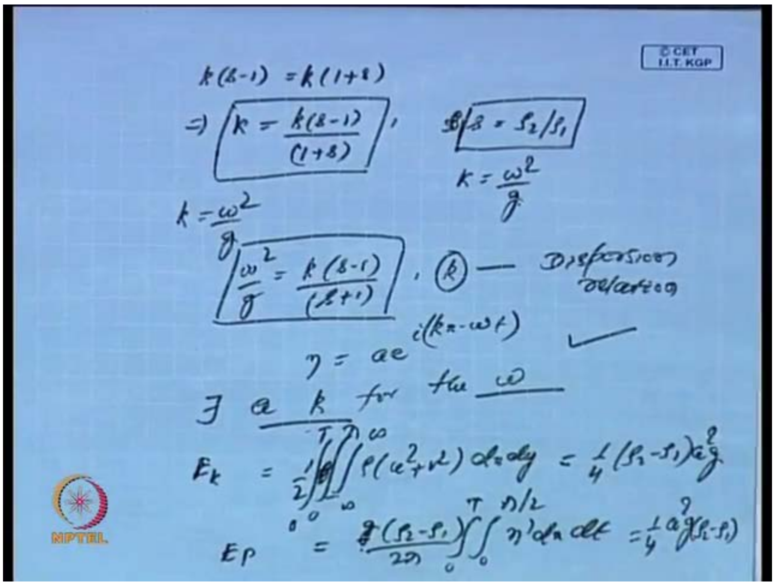


That will give me basically k into s minus 1 is equal to capital K into 1 plus s, which gives me k is equal to capital K is equal to small k into s minus 1 by 1 plus s, which says that and s is where, s is rho 2 into the power i k x y 1 is into y k x rho 1 by rho 2. This is rho 2 by rho 1, it can be easily seen there and then, once we have k is this then, it can be easily seen that, this quantity is always of k's, this is what the and this becomes and what is capital k, capital k is omega square by g capital K is your omega square by g.

So, what does it says us and we have seen that, in case of a single layer fluid, we have capital K is omega square by g where in case of two layer fluid, where I have not to both the layers are covered then, we are getting omega square by g is equal to k into s minus 1 by s plus 1, whatever it is. Now, this quantity is a quantity which is less than 1 and if it is less than 1 so that means, omega square by g is k. So, that k becomes higher that means, this k is higher compared to the capital K, which is omega square by g.

And in the process, the wavelength of this wave will be much lower than the wavelength of the wave in case of a surface wave. So, here so, the because of the density now, we have we have started assuming, that our eta is equal to a e to the power of i k x minus omega t. And we are about to establish a relation that, this k is a related with a frequency at a wave and one dispersion relation exist, we have a dispersion relation and which exist.

And in the process there exist a k for omega for the for omega omega that means, for each period we have a k, that is a wave number so, that shows that existence of a wave at the interface of the fluid, that is what. Now, in the same way, if I just assume, look at the energy so, if I look at the energy, what will happen to the e k and e k is the potential air in the kinetic energy. And if you look at the e k because, in my depth I have taken as minus infinity to infinity and so, what I will do, I will integrate over u square plus v square that is, half.

This will give me rho half m v square half rho u square v square plus d x t y and then, over at over 1 length wavelength 0 to lambda. If I calculate this then, easily I will get it, that I will get this is equal to rho is a, here the rho is varying, rho is rho 1 and the upper layer rho is rho 2 on the lower layer. And that all gives me 1 by 4 of course, I have to integrate if I have to integrate over t so, 0 to capital T< Tt is the period of the wave so then, I will get it 1 by 4 rho 2 minus rho 1 into a square g.

In a similar manner, if I look at the potential energy of this wave, I will get it 1 by lambda, rather again I will apply the same formula. First I will take over the lower layer minus the fluid over the upper layer by the using the definition of the potential energy and I will get g times rho 2 minus rho 1 by 2 lambda and to 0 to lambda by 2 eta square d x and d t, again 0 to t that will again give us 1 by 4 a square g into root 2 minus rho 1.

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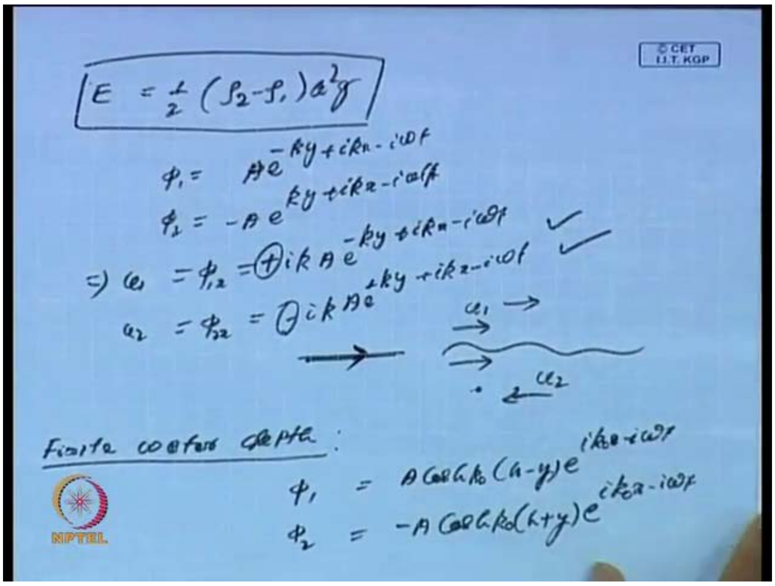

And if you combine these two then, the total energy will give us, e will be give us 1 by 2 rho 2 minus rho 1 into a square g. So, this is the total energy, which will propagate energy density, that will be the other interface of the two layer of the fluid. Now, with this understanding, if I just look at what will happen to the velocity of the waves, I have already given phi 1, my phi 1 is a e to the power minus k y plus i k x minus i omega t and my phi 2 is minus a e to the power k y plus i k x minus i omega t.

So, what will happen to e works e 1 that is, phi 1 x and that will give me, i k a minus k y plus i k x minus i omega t and u 2 would be phi 2 x that is, minus i k a e to the power minus k y plus k y plus i k x minus i omega t. So, what does it means, here if you look at the direction of propagation because, this is a plus sign and that is a minus sign that means, as if the wave that is propagating at the free surface. Although, they are the wave is propagating in this positive direction but, this sign shows that, as if the particle is moving in the the wave particle are moving in the opposite direction.

So, the wave propagation in both the cases, in both the layers are in the same direction, wave is propagating in the same direction. But, the wave particle that is moving that means, u 1 is propagating in positive direction whereas, u 2 is propagating in the negative direction. So, that is another result we get it from here, that is in case of a single layer fluid having an two layer fluid when the lid is covered. Now, the same concept if I go, that I let us look at what happen in case of finite water depth, in case of a finite water depth, it can be easily seen that phi 1 because, my depths are same.

On my phi 1 of this form, a cos hyperbolic k naught h minus y e to the power i k naught x minus i omega t and my phi 2 is minus a. It can be easily seen that, cos hyperbolic k naught to h plus y e to the power i k naught x minus i omega t.

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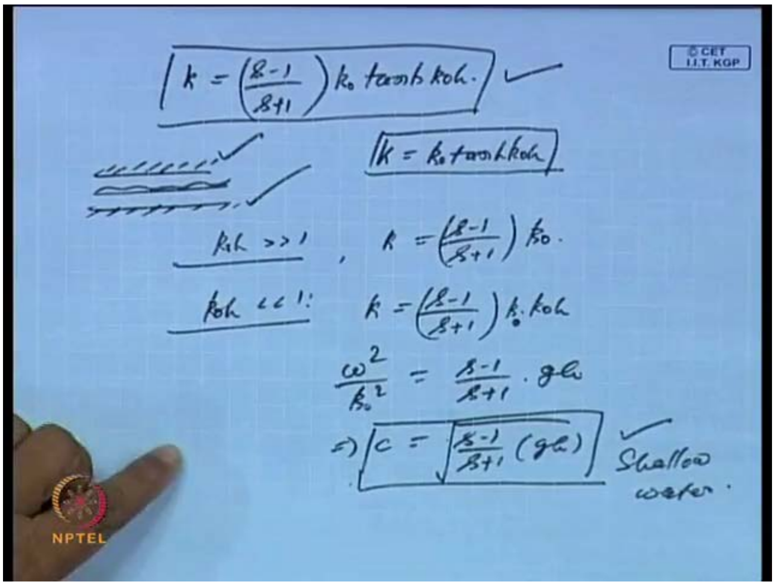

And if they are the then, if I utilize that the interfacial condition, I will get capital K as again s minus 1 by s plus 1 into k naught tan hyperbolic k naught h. So, this is what I will get it, on the other hand we have seen that, in case of a single layer fluid having a free surface, we have seen capital K is k naught tan hyperbolic k naught h. So here, in the two layers so, there are two layer fluid so that means, the k naught this quantity, is a quantity which is less than 1.

So, automatically k if k is the same then, this will be k naught will be high so, if k naught is large then, that will show as if k is lambda is small. So, the wavelength of the wave that will propagate at the interface will be smaller in length in this case. So, this is also showing the existence of a wave at the interface, in case of water of finite depth when there are two lids two fluid and both are covered by a lid. So, that shows the existence of wave at the interface now again, what will happen in this case, this is also obvious if you take it k naught h is large.

If k naught is large then, K is equal to s minus 1 by s plus 1 into k naught, further if your k naught is small, they are very small. Then then also, the capital K will be s minus 1 by s plus 1 into k naught into k naught h and that will give me k is omega square by g h that is, omega square by k naught square omega square by k naught square equal to s minus 1 by s plus 1 into g h. So, which implies c is equal to s minus 1 by s plus 1 into g h so, this is similar to that relation y (()) of tan.

I have seen that, in case of a singular layer fluid in case of shallow water, you have c is root g h whereas, in this case, when to cover two layer of fluid, I covered by two lids, I interface then, the c becomes this in case of shallow water, this is in case of a shallow water. So, this is very interesting to observe that, in all the cases we have seen that, at the interface, we have a wave.

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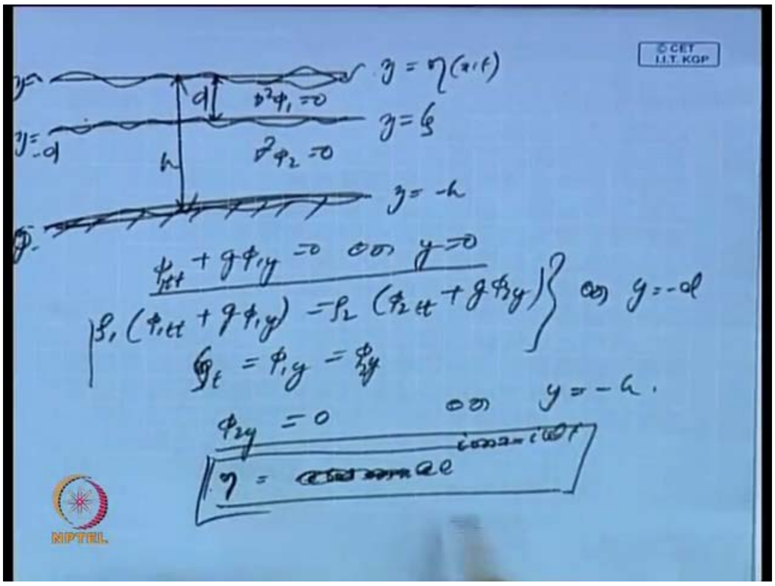


Now, with this understanding, let us see what will happen if I have a I have a both as a free surface as well as the interface, instead of covered by a lid I look at a real ocean where, where the water depth is large. So, let me just have a look at this flow length so, here I will say that, as if I have a free surface here, this is my y is equal to eta and here, this surface I call as y is equal to theta and this is above, I call this bottom as a uniform belt.

And we see the total depth as, we call this the mean free surface here, which y is equal to 0 and here, the mean surface is y is minus d and let me call this as y is equal to minus h. So that means, this distance is h and as this to this distance, if I look at the mean surface I call this as a d. So, in this case what will happen, like in case of a singular fluid, I will have del square phi 1 is 0 here and del square phi 2 is 0 here and I will have phi 1 t t plus g phi 1 y is 0 on y is equal to 0.

And I have rho 1 phi 1 t t plus g phi 1 y is root 2 phi 2 t t plus g phi 2 y plus eta t where, eta t is phi 1 y is phi 2 y and that is, these two conditions are satisfied on y is equal to on the main interface y is equal to minus d and you have phi 2 y because, this is a rigid belt. So, phi 2 y is 0 on y is equal to minus h so, this is the way, the boundary condition is formulated. So here, I am assuming that, I have a free surface and I have a interface so, I have not gone to the detail.

Because, this is already a well known formula, which we have derived free surface boundary condition in the presence of atmosphere, when the surface is open to that atmosphere. And we have seen the just the previous just just now, we have derived that the interface these two conditions are satisfied, that the linearized interface conditions. And if you look at the solution form, solution form will look like if I look at a wave eta, a is equal to a cos m x minus r cos rather portrait in this way, a e to the power i m x minus i omega t.

Then, will have 2 phi suppose, I start with a wave eta if I say that, I do not know exactly I say suppose, I have a wave eta that is exist sorry.

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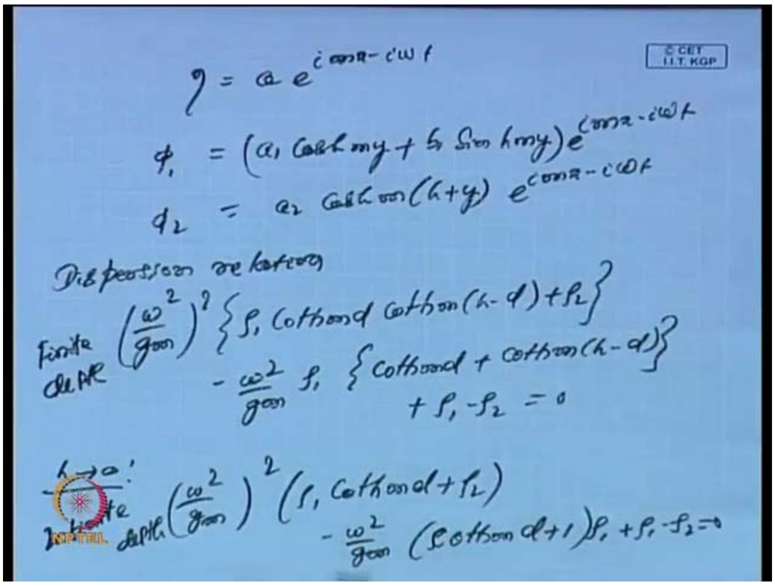


Then, what will happen, then if my sorry if I put it eta is equal to a e to the power i m x minus i omega t. Then, what will happen my phi 1, I can have a phi 1 of this form a 1 cos hyperbolic m y plus b 1 sin hyperbolic m y into e to the power i m x minus i omega t and my phi 2 will be a 2 cos hyperbolic m into h plus y e into the power i m x minus i omega t.

So in fact, this will be have wave phi 1 and phi 2 and again, I am not again repeating many steps I am leaving, again it can be seen that, if you look at the free surface and interface condition phi 1 and phi 2 will satisfy, you will come across a dispersion relation. And you know, in this case the dispersion will be less or will be a little complex and that will be of this form, omega square by g m square into rho 1 cot hyperbolic m d into cot hyperbolic m into h minus d plus rho 2 minus omega square by g m into rho 1 cot hyperbolic m d plus cot hyperbolic m h minus d plus rho 1 minus rho 2 is equal to 0.

This will be the dispersion relation in this case and here, we have seen on like in case of a single layer fluid or two layer fluid having a common interface where, the lids are covered we have seen, the dispersion relation was a relation omega square by g m. But, in this case, the dispersion relation is a quadratic in omega square by g m because, here there is a square term, here there is a linear term in the omega square by g m. So, it is obvious that, if you solve this for omega square by g m then, we will get 2 omega square by g m.

And let me call this and again if you say, look at a infinite depth when h is h will be large then, in that case this will be simplified to omega square by g m square into rho 1 cot hyperbolic m d plus rho 2 minus omega square by g m into sorry this is cot hyperbolic m d plus 1 into rho 1 plus rho 1 minus rho 2 is equal to 0. So, this is for finite depth, this is for infinite depth but, in both the cases. Because, we have one free surface, one interface, in the both cases omega square by g m is a quadratic equation, n omega square by g m the dispersion relation is a quadratic equation.

We suggest that, as if and in this case of infinite water depth, when h is a large in case of infinite water depth.

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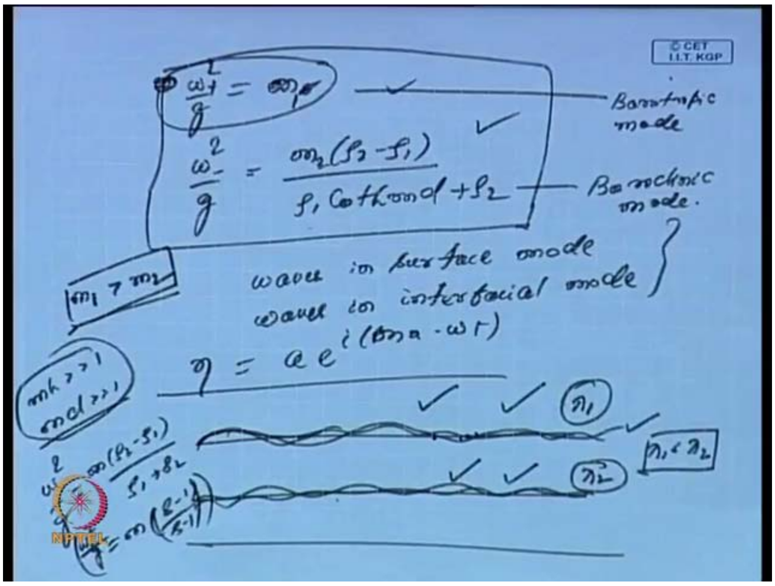


We will see that, that two relations will be coming out omega square by g is equal to rather, I will call this one was as m 1 and or this as m and another thing omega square by g minus. Because of the two quadratics, this is call it m into rho 2 minus rho 1 by rho 1 cot hyperbolic m d plus rho 2. So, these in case of the infinite water depth, this can be factorized in this form and which suggest that, which is a gives a clear idea that that, omega square by g, it was a quadratic which has been factorized into these two form and this is m.

Then, which shows that and if we look at this one, this is the similar to that omega square plus by g g m, this is similar to the infinite depth dispersion relation that is, if I call k as the m square by g as the wave number then, omega square by g is equal to m is the (()). So this is that means, such a way it exists every where, it can also be seen in that, this omega square by g is equal to m is one of the, this m wavelength this is a wave which propagate everywhere on the other hand.

And this one, this relation will refer to the waves the corresponding m call it m 1 and here the corresponding m we will call it as m 2. And here, m 1 will refer to the mode in surface called the surface mode and m 2 will refer to the mode wave mode in interfacial mode. And sometimes, we call it is waves in surface mode and then, the second relation will give us the waves in inter interfacial mode so, we have two mode. And in the process what happen that means, I started with a common eta is equal to a e to the power i times m x minus omega t.

And I am finding, I have the existence of two such modes and which shows that, there are two waves, one wave is propagating at the surface, the other wave is propagating at the interface. So, that is in case of a two layer fluid, have two wave which will exist, one wave will propagate at the free surface, the other wave will propagate at the interface. Again if you look at this relation then, easily we can see that, that m 1 because, this is omega square by g, it is m 2 times this quantity and this will be…

So, that will give us m 1 is greater than m 2, it can also be seen from here so, in general m 1 will be greater than l 2 m 2 and then in fact, another point to be noted, that this refers to the barotropic mode. Oceanographic term, it is called barotropic and this term refers to as to baroclinic mode, waves in the surface mode or surface mode, this is surface mode, this interfacial or internal mode interfacial mode or internal mode. So, with this understanding so, I have when I have a wave at the surface, here I have a wave at the interface and again I have m 1 is m 2.

If m 1 is m 2 that means, waves wave number in surface mode is greater than the wave number. So that means, the wavelength here will be less, corresponding lambda 1 will be less and here, lambda 2 will be the this lambda 2 will be more that means, the lambda 1 is less than lambda 2. Wavelength of the waves in surface mode is less than the wavelength of the waves and this can be easily verified in the case of finite water depth.

Again if you look at this, if my m d is much greater than 1 that means, this distance increases then, I will have omega 2 square omega minus square by g will be m into root 2 minus rho 1 by rho 1 plus rho 2 and that is same as m into s minus 1 by s minus 1. So, even if so, when the we have just in case of a two layer fluid in case of a two layer fluid having covered by lid we have seen, this a relationship of this type omega square g is equal to m into s minus.

But here, when the two fluid that means, are wide apart from each other, have a the interface is wide apart from each other in free surface mode. So, here we have waves in the free surface and then, there is a wave in the at the interface. But, the wave at the interface behaves as if, this is whether it is a rigid surface or a free surface, it is immaterial when we have both m h is much greater than 1 and m d is much greater than 1.

So, we will have an interfacial wave when the two waves two surfaces are away from each other then also, when the interface moved, the interfacial waves will be independent than the surface waves. And here, the surface wave whether it is a there is a it is covered by a lid or it is not covered, if it is covered by a lid then, not be any wave. But, when the distance is large, if it is having a if it has a t surface then, there will be a wave propagate and another wave will propagate here.

So, that is how, two waves will be propagating at any point of time when we have two, there are two layers of fluid having a surface and interface this is what, I want to highlight here. Now, the same concept we will again see that, what will happen if we will look at the shallow waters so, we have already seen in case of shallow water.

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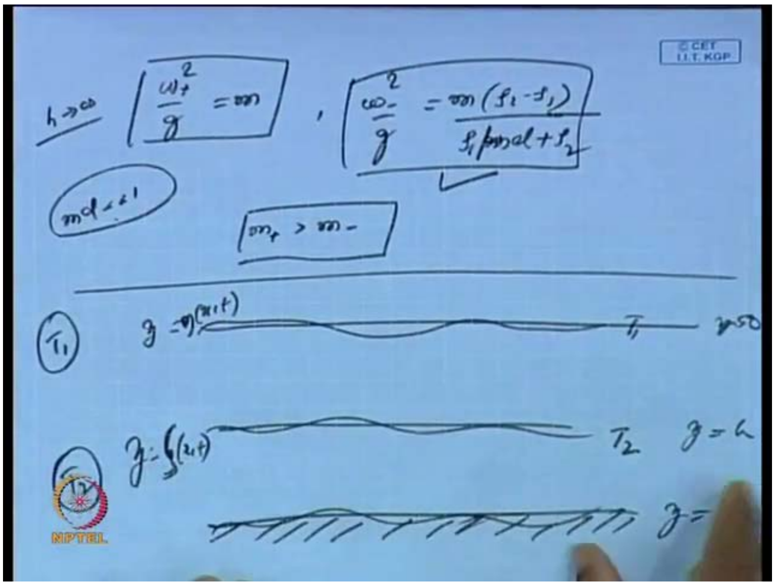

A similar relation will hold good because, in case of shallow water we have already seen omega square plus by g, I have looking at same case when h is infinity but. I am look at m d. M d is less than 1 then in this case, omega 2 square that is, omega square minus g is m into rho 2 minus rho 1 divided by rho 2 rho one, rather rho 1 plus cot hyperbolic m d, cot hyperbolic means it will be cos by sin 1 by 1 1 m d rho 1 m d plus rho 2 sorry rho 1 cot hyperbolic m d rho 1 by m d plus rho 2.

So, in this case also, we will see that, if you simplify this further so, this will be give us again a wave when the two layers because, nothing much will change again. Again it will give us a wave when m d is also less than 1 and so, these are the in all the cases, we will see that, m plus is greater than m minus. And that is why, that wave number of the waves in the surface mode is higher than the wave number of the waves is in the interfacial mode.

Now, with this understanding so, we have just a few days back, we have in the one of the lectures we have talked about the capital at the gravity waves. So, what will happen if I have a I have two layer fluid having a free surface, there is a surface tension here and I say that T 1 and there is a interfacial tension, I call it as t 2. And if I say, y is equal to 0 is y is equal to eta is the free surface and y is equal to eta is the interface and let me call this mean free surface, as y is equal to 0.

And this main interface as y is equal to h and the bottom, I call it as a uniform band and if the bottom is y is equal to capital H, I have taken here the downward direction as the positive direction. Then, in such a situation what will happen, here I am looking at the waves in the presence of surface and interfacial tension. So, T 1 is a surface tension, T 2 is the coefficient of interfacial tension. Then, in that case what will happen.

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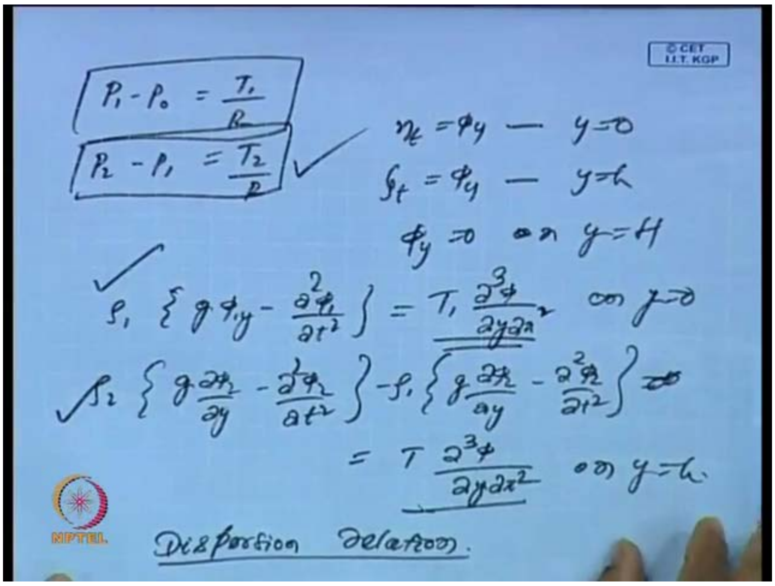


My P 1 minus P 0 is equal to T 1 by R and P 2 minus P 1 is equal to T 2 by R, this is a two condition will come and again that is the again I will have I will have eta t is phi y on the surface have g eta t is equal phi y at the interface, this is linearized condition, this will be on y is equal to 0, this will be satisfied on y is equal to h. And I will have phi y is 0 that is, on y is equal to capital H then, from these two conditions, I will get the if I combine this with the gravity, again it will give me the interfacial conditions and the surface condition if I combine this with the gravity.

P naught is the atmospheric pressure, P 1 is the hydrodynamic pressure on the upper layer of the fluid and P 2 is the hydrodynamic pressure on the lower layer of that fluid. Then, I will get in these case, my interfacial condition as rho 1 into g phi 1 y minus del square phi 1 by del t square is equal to T 1 del cube phi by del y by del x square. That is, on y is equal to 0 on the mean free surface, under the mean interface I will have rho 2 g del phi 2 by del y minus del square phi by del t square minus rho 1 g del phi 2 by del y minus del square phi 2 by del t square.

And that will give me 0 sorry it will not be 0 in this case, because of this condition and that will give me T 2 by R and then, if I linearize, look at the linearized our condition that will give me del cube phi by del y del x square. So, this becomes my condition on y is equal to h so, this is the free surface condition in the presences of surface tension and this becomes the interface condition in the presence of surface tension. And again, I have the bottom condition as I had told, at the bottom we have phi y is 0, that is the normal velocity is 0.

So, in this case what will happen, if I look at the dispersion relation, again I will have a dispersion relation which will exist but, this dispersion relation because of the higher order, this term, this term and this term it will have six roots and that six roots will give me six solution. Out of the six solution, two of the solutions are associated, look at the dispersion relation and the dispersion relation will be very complex. And that dispersion relation will give us six roots and out of the six roots, we will have two roots will be real, and four roots will be complex.

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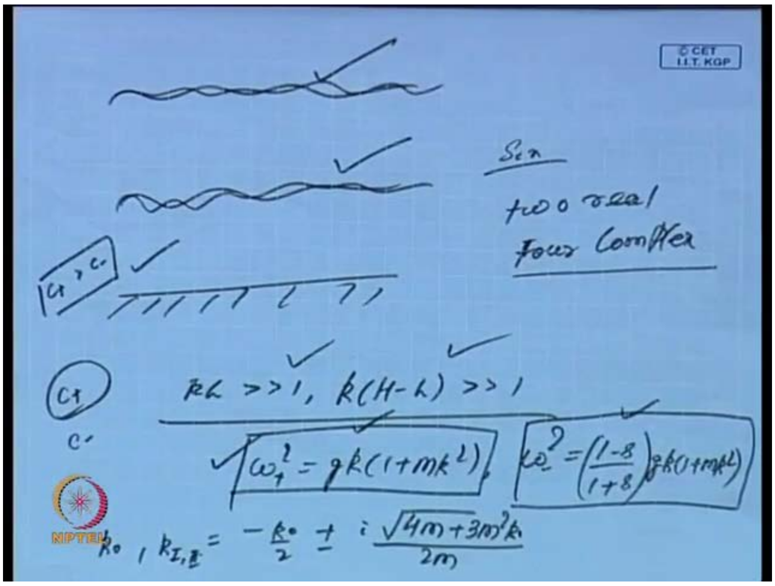


And those six roots will refer to three waves of the interface and three wave three wave number at the interface due to the waves at the surface and three wave number will relate to the waves due to the waves at the interface. So, in that case and out of the six, two of them will be real, two real and four complex roots. So, there will be one real wave, that will propagate at the free surface and another real wave that will propagate at the interface.

And in fact, it can be seen that, in case of a deep water when k h will be much greater than 1 and k into h minus h is greater than 1. That will lead us to a situation where, omega plus square here, g k into 1 plus m k square and omega minus square will give us wave, which will be of the form 1 minus s by 1 plus s into g k 1 plus m k square and that will show, that is only when these two relations hold. That means, again this will relate to the wave as if, in the presence of surface tension in a single layer fluid.

On the other hand this finally, both the results will give us this is most simplest problem for the two layer fluid in the presence of interfacial tension. And this will refer to the waves in the surface mode, this will refers to the waves that is propagating at the interface mode. And again, if you look at the real root from this wave, we will get it will have three roots, one is the real root k naught, the others are k 1 and k 2 and this complex roots will be k 1 will be minus k naught by 2 plus i times 4 m plus 3 m square k naught divided by 2 m and then, again another one will be minus.

So, this is k I, k I I that will be associated those wave and these are the complex roots similarly, in case of interfacial waves, we have we can easily get from this we can get c plus that means, the phase speed in the case of surface mode. And we will get c minus and it can be further seen that, where the phase speed of the surface mode that means, c plus is greater than c minus. That means, now phase speed, the wave propagating at the free surface is much higher than the wave that propagate at the interface.

So, that is another relationship so, with this, again there are several other results which can also be obtained in case of, if you look at a in case of shallow water. In case of shallow water, again we can see that, same in the case of shallow water particularly, in the presence of interfacial tension.

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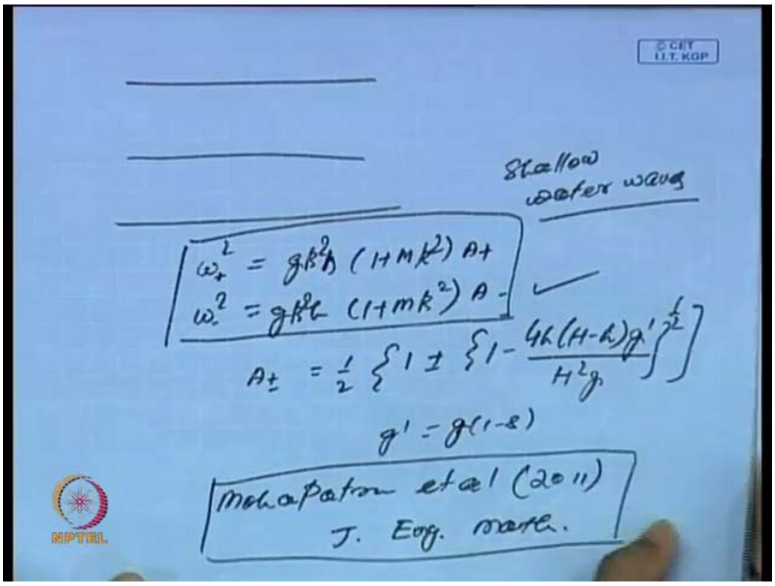


In case of shallow water also, we can get that, omega plus square that equal to g k square h 1 plus m k square and into some constant A plus and omega minus square g k square h into 1 plus m k square A minus. And this A plus, A minus where, A plus will be A plus minus will be 1 by 2 1 plus minus 1 minus 4 H into H minus h into g prime divided by H square g to the power half where, g prime is equal to g into 1 minus s. So, this is a kind of relational it comes.

So, it is a little complex what one can get this kind of relation that, when you have dealing with shallow water waves in the presence of surface tension. So, these relations I have not deriving the present context but, this will be very interesting to observe and it can be these all these results can be obtained from a paper by Mohapatra and it is some of the one of the recent paper in 2011, in journal of engineering mathematics. Some of the results can be derived where and many other results associate with the upper gravity wave motion in two layer fluid can be obtained from this article.

And what here we are looking into, what we have concluded today, that in case of a two layer fluid in case of a two layer fluid we have seen that, there are two waves that propagate. One wave propagate at the free surface, the other wave propagate at the interface and often we call them as, waves in surface mode or barotropic mode and the wave that propagate at the interface, we call this waves in internal mode or baroclinic mode.

And in fact, this is one of the very important factor particularly, in the ocean in the norogen force, it has been observed for a long time that, ships used to come across a kind of resistance when it used to pass through a particular this norogen force. Where, the resistance comes additional resistance comes often due to the presence of these two layer fluid and that one is because of this surface layer, the other is because of the interfacial layer.

The interaction with the two layers provides maximum enormous amount of resistance, which creates a obstruction, which provides a resistance of a nature and which damage the ships or sometimes it creates more obstruction and the pilot, a captain of the ship faces more resistance while while travelling in across these fords. So, that as a that phenomenon remain a dead water phenomenon for a long time until in the mid 19 th century, when one of the oceanographers pointed out that, this kind of resistance is due to the presence of the internal waves.

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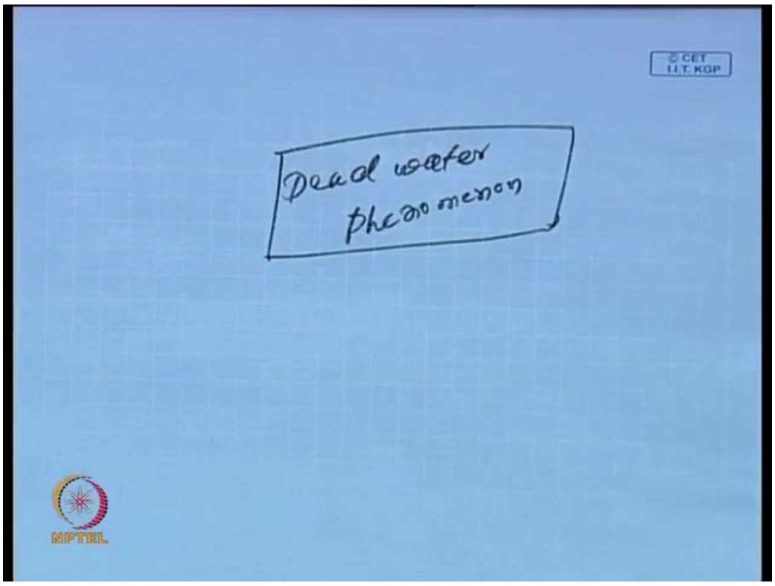


And otherwise it was known as a before that, it was known as a dead water phenomenon and there are several other applications of this because, particularly this internal waves are plays a significant role in many other studies. Particularly, when we see that, in oil spill in case of oil spill, often this even if the results are associated to the interfacial tension and the surface tension, sometimes surfactant layers are in the surfactant layers some certain surfactant in the interfacial layers. Surfactants are spread to extract the spilled oil on the at the interface from the surface of the water by.

So that because, the oil particularly cling to this surfactants at the interface and then, it will sometimes easy to extract those oil those surfactant particle where, the oil to reach the oil particle is attached. And that is one of the mechanism sometimes used in the oil spill recovery in the two layer fluid by using increasing the surface tension at the interface. And there are other application particularly, which oceanographers worry about that change in the temperature and the mixing of water, of the hot water with the cold water and then salinity, the change in the salinity all these things.

But here, in this case, we have considered as if the fluid is, there is a sharp density change in the density. But, on the other hand, when there is a there is a stratified fluid where, the density is there is no sharp change in the density rather, continuously there is a change in the density, the phenomenon is more complex. And but, this understanding about the, when there are sharp change in density, it gives a good understanding about the internal waves that propagate at the interface of the fluid, when there is a free surface also, with this only stop here.

Thank you.