Dynamics of Ocean Structures
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Module - 1
Lecture - 20
Orthogonality of modes

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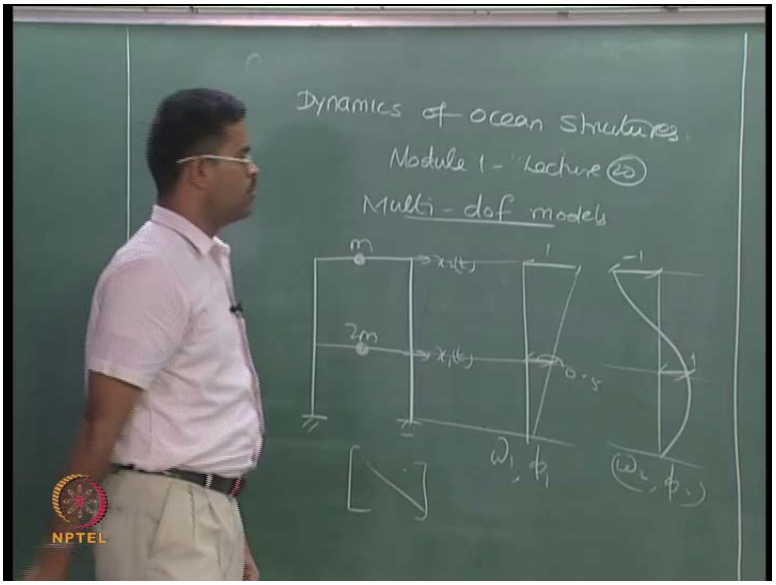

So, in the last lecture we have discussed about the orthogonality of the modes. We have an example here where the m 1 and m 2 are concentrated at the same point where the displacements are measured. Therefore, the mass matrix must obviously diagonal; the half diagonal elements will become 0. And we have worked out the natural frequency and mode shapes of this model using a classical Eigen-Solver theory, where I said, the omega 1 and omega 2 are identified as 1 and 2 after I found out phi 1 and phi 2, because phi 1 and omega 1 becomes a pair. So, phi 1, where has got 0 crossing, in my mode shape will tell me, it is my first fundamental frequency. Therefore, omega 1 becomes 3 point something and omega 2 becomes 6 point something where phi 2 having a 1, 0 crossing, here it was second mode and first mode. So, we already had the advantage of explaining, that why modes need to be orthogonal. If the modes are orthogonal, the reciprocal theorem can be directly applied to analysis of dynamic analysis structures as an interpretation.

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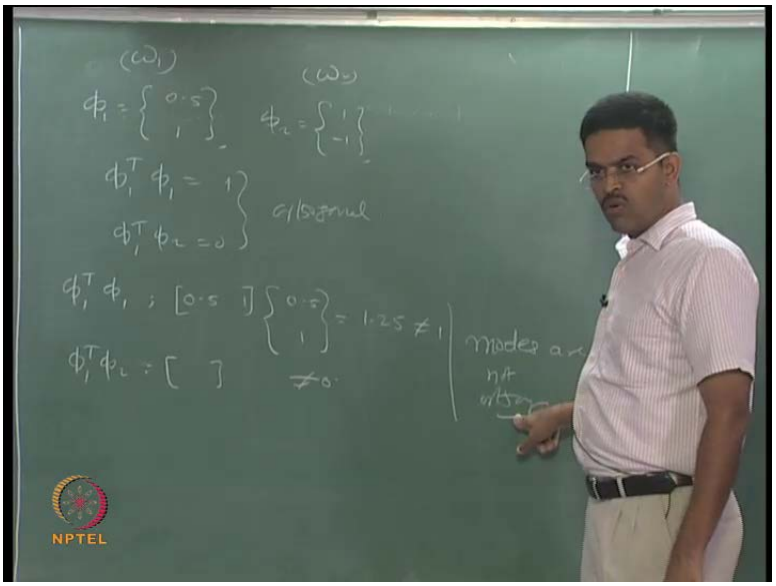

Therefore, to qualify the modes to be, let us say, phi 1, in my case, is 0.5 and 1 and phi 2, in my case, is 1 and minus 1, which are relative displacement of the mass points when they are vibrating at a specific frequency, namely omega 1 and omega 2 physically. When the model is subjected to vibration of omega 1 radians per second, the mass m 1 and mass m 2 will get relatively displaced in this manner. If physically the model gets vibrated at omega 2, the mass points 2 m and m will get physically relatively displaced in this manner. So, mode shape is a physical representation of relative displacement of mass points at specific frequency of vibration. So, they are the pair at a specific frequency, our specific motion, they are unique pair. So, I must know whether they are orthogonal or not, there are advantages here. So, we said, that if phi 1 transpose phi 1 is 0, sorry is unity and phi 1 transpose phi 2 is 0, then I can call this as orthogonal.

So, yesterday we gave an example to you to check whether they are orthogonal or not. So, quickly let us do this. Let us say, phi 1 transpose phi 1 in this case, so 0.5 and 1 of 0.5 and 1, which is 1.25, which is not equal to 1. Similarly, phi 1 transpose phi 2 can be all the same executed. We will see it is not going to be 0. Therefore, the nodes are not orthogonal, I want to make it orthogonal because there are advantage making them orthogonal, will use this in my analysis in the second module later. So, I want to make them orthogonal.

There are two things, which generally people get confused, one is static decoupling of equations of motion, other is normalizing the modes. Static decoupling is the way in which you select your displacement coordinates. We saw examples in the previous cases where if you select the displacement coordinates of the point where the masses (( )), masses becomes diagonal. Sometimes in the one example we saw masses not diagonal, stiffness became diagonal. So, it is what is called static decoupling of equations of motion, whereas normalization is having a different application directly in dynamics. We will see that later. So, if you want to orthogonalize the mode, which is otherwise non-orthogonal, I call this process as normalization of modes or the resulting modes are called as weighted modes.

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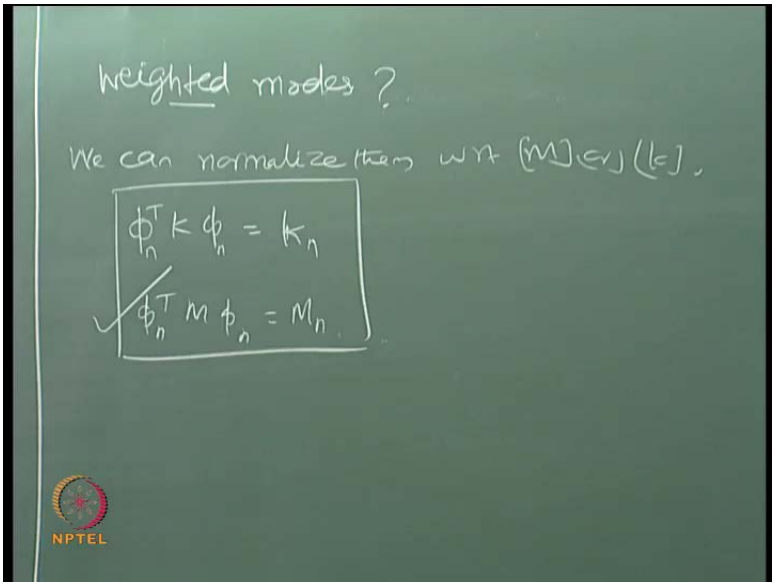

Now, how to apply weighted modes because they are not weighted modes now? Weighted modes are nothing but orthogonal to each other, so we can normalize them either with respect to mass or stiffness. On the other hand, let us say, phi transpose K phi of nth mode should be 1 or phi transpose m phi can be 1 multiplied by k, that is, K n. So, I can use either of this procedure to normalize the modes. We will pick up this in this case and demonstrate it, how I can get the weighted modes for this example.

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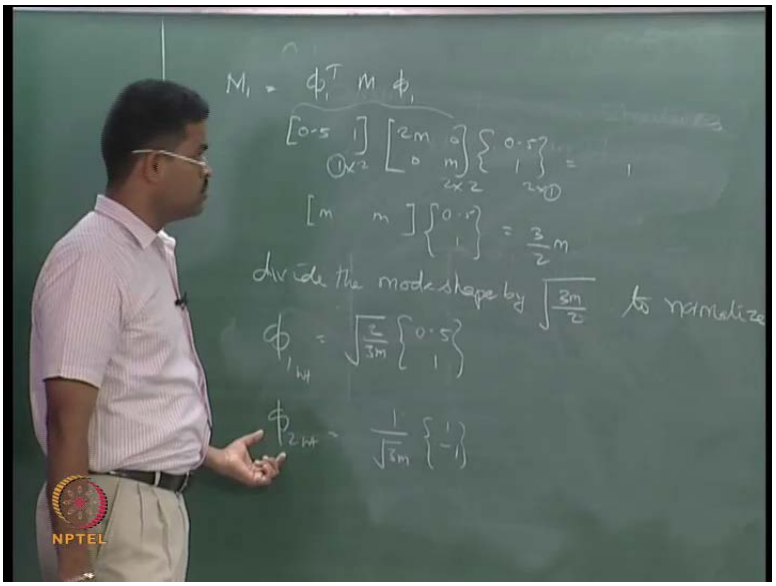

So, let us say, M 1 will be equal to phi 1 transpose m phi, so that is nothing but 0.5, 1; m is 2m, 0, 0, m of 0.5, 1. This is 1 of 2, this is 2 of 2, this is 2 of 1, ultimately you get only one value. So, let us first multiply these two terms and see what happens. So, I get m and m, then multiply this with this, I get one point for m or 3m by 2, is that ok. Now, divide the mode shape by square root of 2 by 3 m to normalize. So, weighted modes, phi 1 weighted will be, phi 1 weighted will be root of 2 by 3 m by 0.5, that is, divide the modes by, divide this by this, so I get root 2 by 3.

Please make the change, divide the square root of this value, all the mode, mode shapes here I get this. Similarly, can I get phi 2, same way can I get weighted modes of phi, so it is becoming root of 1 by 3m of 1 minus 1, is that ok. And even written as 1 by root… It is one and the same, and then we check whether they are really orthogonal now. I have normalized them respect to mass matrix. So, try to see, are they really orthogonal now.

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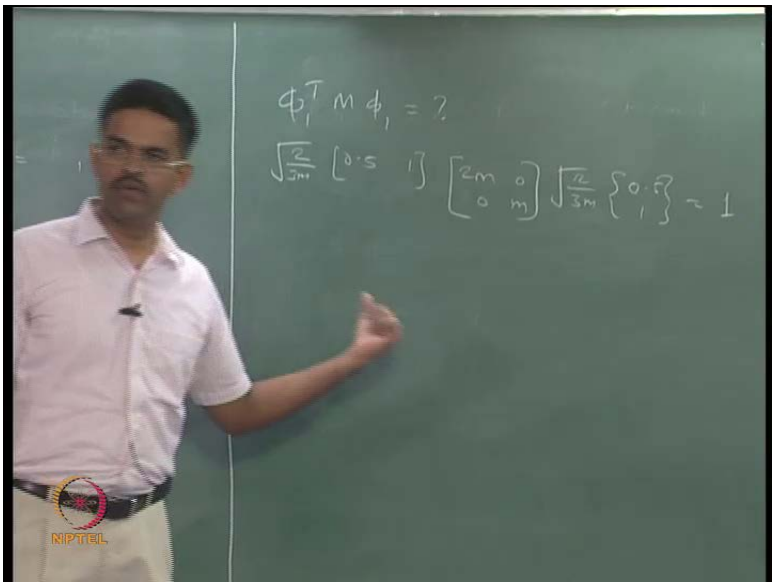

So, try to check whether phi 1 transpose m phi 1 is becoming unity. So, I have got phi 1 and pi 2. So, root of 2 by 3 m, 0.5 and 1, so it will become one check. So, the modes are now normalized, I call them as weighted modes. We will use these weighted modes later for dynamic application. We will talk about that later, but how to normalize the modes, we should know the procedure. You can also do it with respect to stiffness matrix. Any doubt here?

So, what we now know are the following. For a given two degree or single degree or for example, multi-degree, I must know how to write my equations of motion, I must know what would be the mass matrix and stiffness matrices form from the model, I must know how to work out the natural frequency or the frequency of vibration in the mode shape associated with the vibration frequency for all the possible modes for a given model now.

For two degree the classical Eigen-solver procedure is simple. When you go for higher degrees, for example, ocean structures, we have got higher number of degrees of freedom in the given model. It becomes very tedious for us to do a classical Eigen solver procedure for finding out omega because sometimes what happens, depending upon the nature of K and m matrices, you will not be able to even get the classical Eigen-solver problems at all getting done in the programs. So, you must have some shortcut methods, some approximate techniques with the help of which I can find omega phi quickly.

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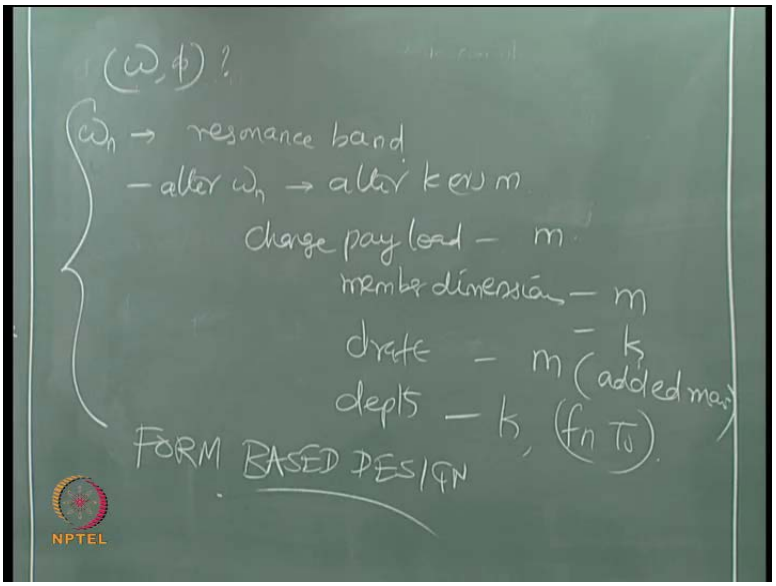

Now, the question is why do I know omega and phi? We already answered this yesterday. I must know the natural frequency of the system because I will check for the resonance band. I can always alter omega n, it means, alter k or m, it means, change the pay load, that is what I do with m. Change the member dimensions, that is, what I do with m and k, change the draft, that is what I do with m. Because added mass change the installation water depth, that is what I do with k, because k will be now a function of T naught theta tension and so on. So, all will be actually form based design. So, I am interested in knowing omega and phi for a given system. So, if it is a multi-degree, how do I do this omega and phi quickly?

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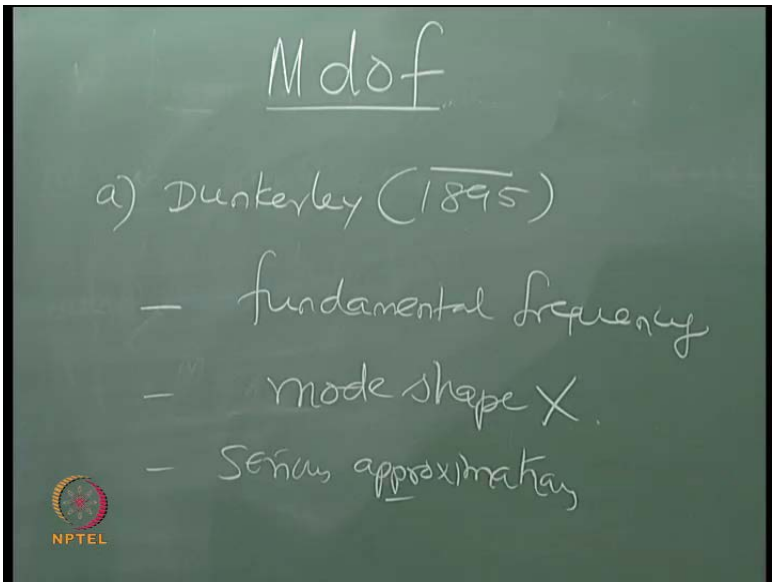

So, for multi-degree freedom system problems, I call this as Mdof, multi-degree of freedom systems. Degree of freedom is representing the displacement coordinates, which are independent. It is nothing to do with the mass points concentrated at all. So, there are many methods available in the literature, the foremost and principle and simple method is given by Dunkerley, Dunkerley's method. This method was proposed by this researcher in the year 1895. So, I want you to throw light on this year of invention.

This method was much older before the computer programs were in place, before computers were used for scientific computation. Dunkerley proposed the method for finding out omega and phi for multi-degree of freedom systems. So, obviously, the system must have been simple. It should be easy for doing hand calculation, that is the advantage of the system, but the greatest merit of the system is, even now I will demonstrate here in the lecture. I will work out omega and phi using scientific methods.

The method of result what you get from Dunkerley closely match with the results, what we have from scientific methods, even now I will show you that here. The method is still valid and demanded because this method gives you an approximate value only of the fundamental frequency. So, the limitation of method is, will give you only the fundamental frequency. Fundamental frequency is supposed to be the lowest frequency. It will not give you the mode shape, it will give you only the frequency, with of course,

some serious approximations; there are serious approximation given. Let us see this algorithm first.

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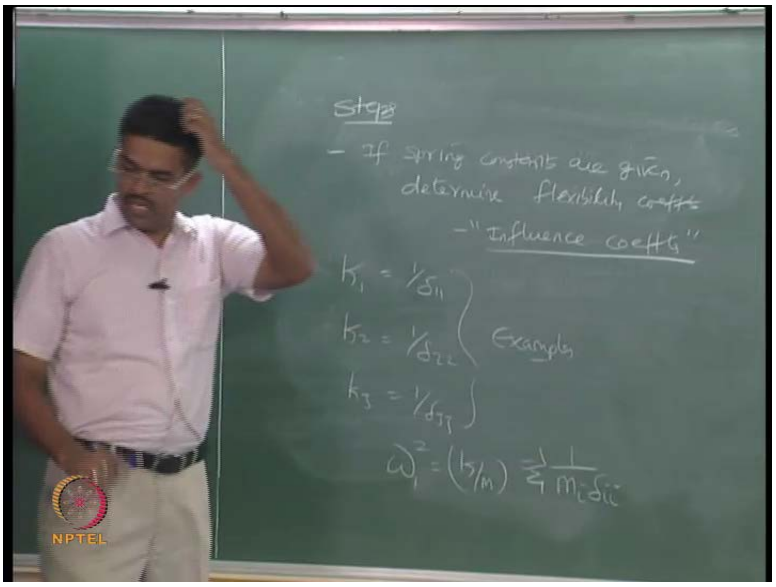

So, Dunkerley proposed the following steps. If spring constants are given, determine the flexibility coefficients. It was a first step, he named this as influence coefficients, that is, the name given by researcher saying, that the flexibility coefficients are otherwise called as influence coefficients. So, for example, if you have k 1 as the spring stiffness given, 1 by delta 11, k 21 by delta 22, k 31 by delta 23, are examples of finding out the influence coefficients when spring constants are given. Now, how do we simplify this problem for a multi-degree? What he says is, omega 1 square is simply k by m, k is 1 by delta. So, what he said simply is, 1 by m i delta ii sum.

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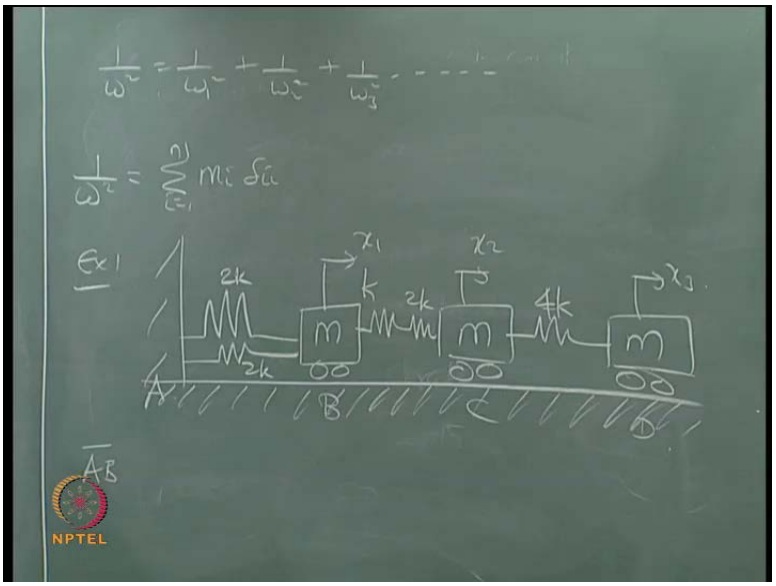

So, if you want to find a 3-D model, for example, let us say, 1 by omega square should be 1 by omega square 2 square and 3 square and so on. It is an approximation made by Dunkerley, which can be applied to multi-degree and he says simply, is sum of m i delta ii will give me omega square. So, we will take up simple example and see how this can be demonstrated, we will take up an example.

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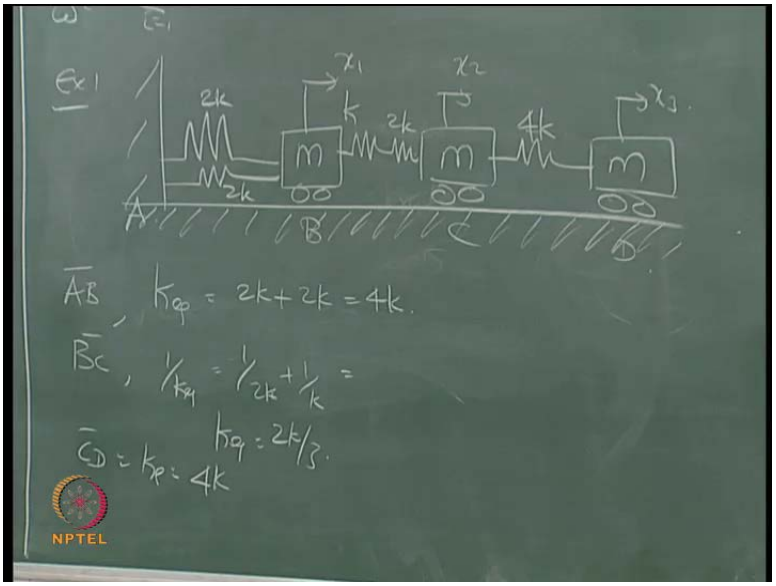

So, degree of freedom one, degree of freedom 2… So, let me name these segments as A, B, C and D for segment AB. I can work out equivalent stiffness. So, k equivalent is 2k

plus 2k equals to 4k. Similarly, for the segment BC, 1 by k equivalent, because they are in series, 1 by 2k plus 1 by k, so k equivalent becomes 2k by 3. Similarly, for the third case, that is CD, k equivalent is nothing but 4k. Let us remove this.

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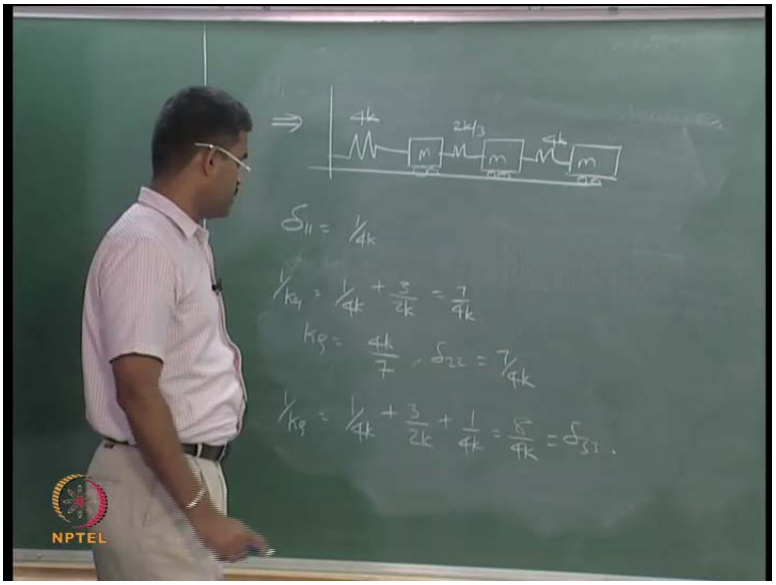

So, the system is now converted to an equivalent system like this and delta 11 is for the first case, is nothing but 1 by 4k. When I come to the second case, these two springs again becomes in series. So, 1 by, K 1, k equivalent again is 1 by 4k plus 3 by 2k, so 7 by 4k of k equivalent is 4k by 7 or delta 22 is 7 by 4k. Similarly, for the third case, these three springs become in series, therefore 1 by k equivalent is 1 by 4k plus 3 by 2k plus 1 by 4k, 8 by 4k, which is nothing but delta 33.

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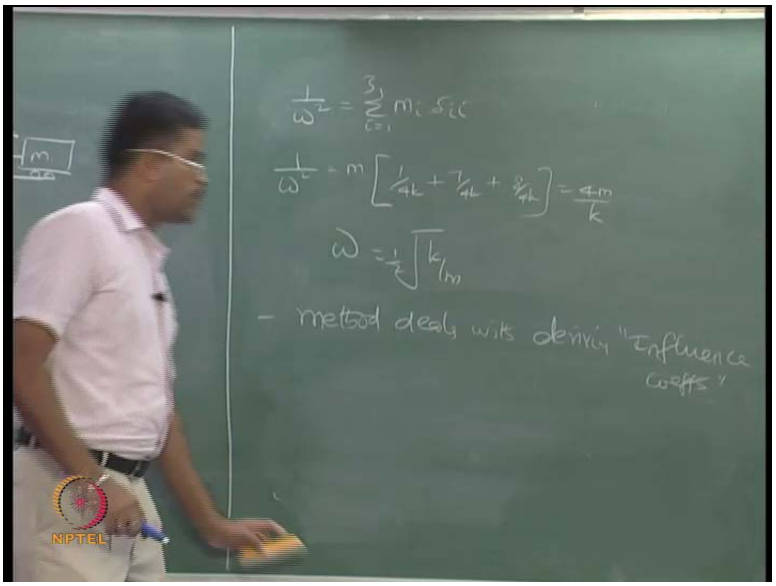

So, 1 by omega square sum of m i delta ii, i varies from 1, 2, 3 here. There are three degrees of freedom here. So, 1 by omega square is going to be equal to m of, because m is common otherwise you can use the multiplier inside, 1 by 4k plus 7 by 4k plus 8 by 4k, so 4m by k of omega is. So, it will give me the fundamental frequency. It will not give me the mode shape, it has series approximations, that it is converting the given system equivalent spring mass system like this. So, the method deals with deriving what we call influence coefficients of the given form. So, one of the approximation by which you can find out the fundamental frequency for a multi-degree, the second method is extension of this, but with an idea to also find the mode shapes.

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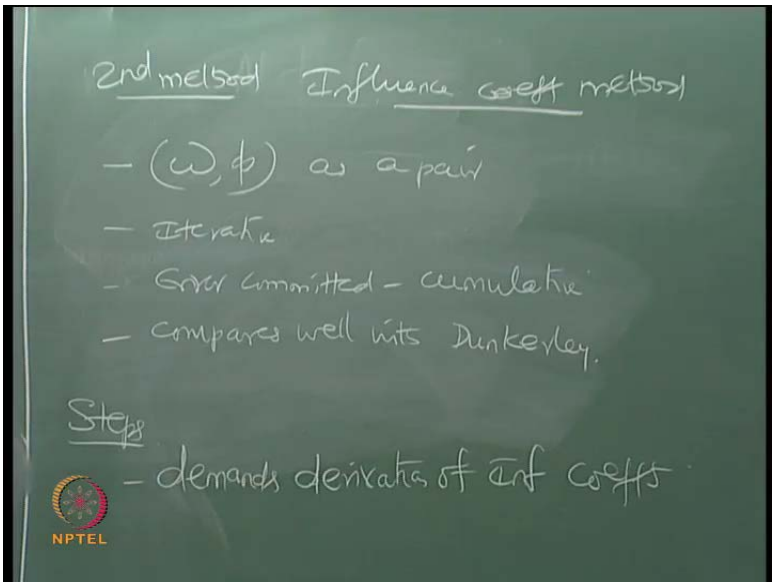

So, the second method is called influence coefficient method. This method will give me omega and phi as a pair. This method is iterative and the error committed would be cumulative and this method compares well with Dunkerley. Let us see what this method is. This method also demands steps, this method also demands derivation of influence coefficient. So, let us derive the influence coefficient first for the next example, which I show here.

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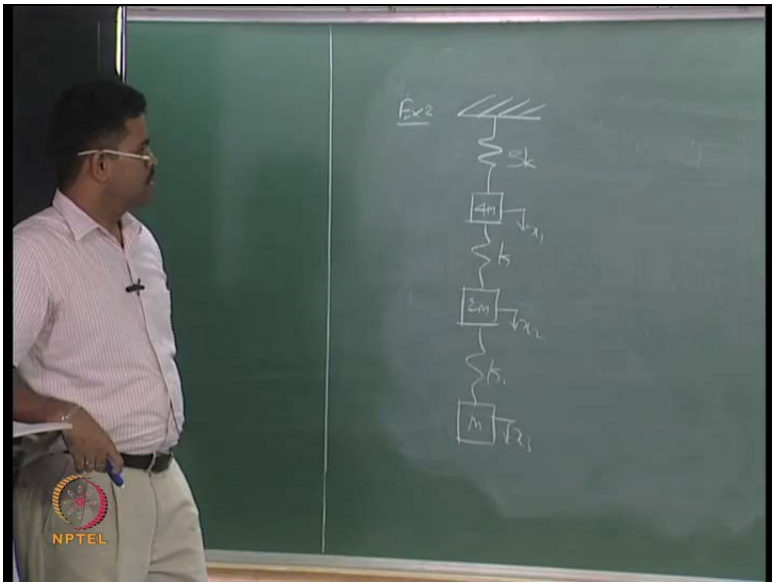

So, I have a problem, which is again a three-degree freedom system, 4m, 2m and m; 3k, k and k; x 1, x 2 and x 3. So, the first step, what I want to do in this problem, is derive my influence coefficient.

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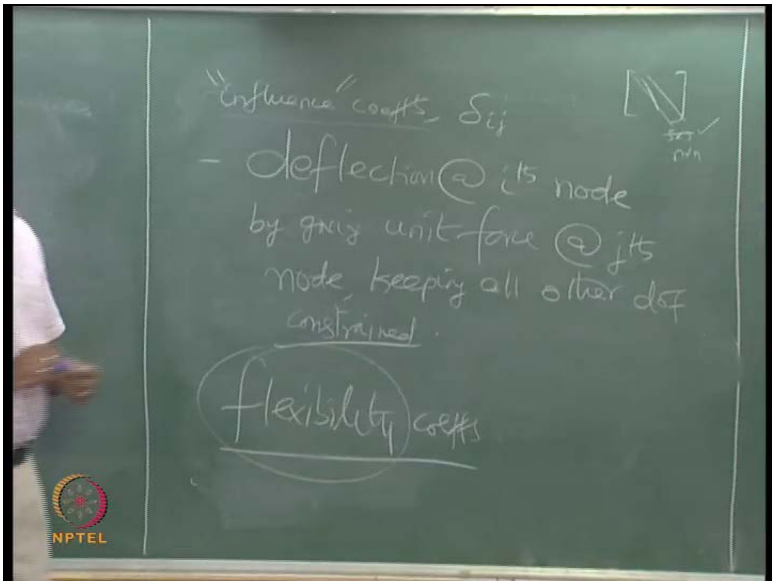

The influence coefficients are indicated as delta ij, which is deflection at ith node by giving unit force at jth node keeping all other degrees of freedom constrained. This definition is the derivation or the explanation is similar to the stiffness matrix. The stiffness says, it is a force responsible for creating unit deflection, but influence coefficient is nothing but deflection, which is caused by unit force. Therefore, influence coefficient is nothing but flexibility.

One may wonder, sir if I can derive the stiffness coefficients or the stiffness matrix already from the given problem and invert it? I can find the flexibility coefficient, why I am interested in finding out the influence coefficient directly? First, inverse of a matrix beyond a specific size itself is a problem. When this method was introduced inverse of a matrix itself was a problem.

For example, if you have got a three by three matrix with symmetry, we have a procedure. If you have got n by n, which is banded, we have a procedure. If you have got n by n, which is a full matrix, which is unsymmetric and non-banded, it is very, very tedious. And moreover, when this matrix keeps on changing with respect to time in dynamic analysis, each time inverting will be a problem because the coefficients by

mistake can also become non-invertible. It is possible, by any mathematical mistake the coefficients can become or the stiffness matrix can get non-invertible, means, this and the analysis will stop at that point. So, but your analysis needs influence coefficient, which requires inversion, which will not execute your omega and pi at all. So, it is always better, that for this method I must derive my influence coefficients, not the flexibility, but they are one and the same.

One may wonder, then why the name of influence has been given, why not flexible influence name has been given? because these are the coefficients, which will influence the arrival value of omega and phi. One, one may wonder, sir, are we going to assume this coefficients because they may influence my result? It will not influence the result; it will influence the number of iterations to reach the result. The number of iterations to reach the result will be influenced, that is why, these coefficients are called influence coefficients. By definition they are nothing but flexibility coefficient. Flexibility is nothing but inverse of stiffness matrix. I know how to derive a stiffness matrix for this given problem from the first principles; we have done it for different cases.

So, we know, but we will not use that technique to form the influence coefficient matrix. We will derive this matrix here and we will follow the same steps for henceforth, all the problems where I want to find omega and phi using this method. That is why this method itself is named as influence coefficient method. I will show you how it is an extension of Dunkerley's method. It is an extension of this method, I will show you, that once we understand the algorithm of this method.

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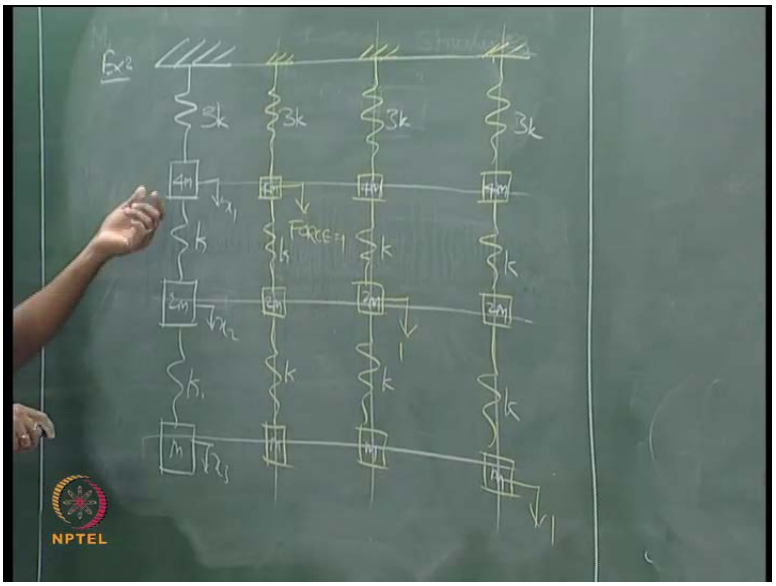

So, let us first derive the influence coefficient for this problem. So, what I should do is, I must provide or excite a unit force at any specific node and find deflections in all these, right. So, let me give unit force in all the three degrees of freedom independently. So, this is my model, I give unit force here, I give unit force here, I give unit force here, that is how I should do. This is 3k, this is k, this is also k. This is 4m, 2m and m. So, let us, tick, take this example, the first case, where I am giving unit force as the first coordinate, the one which is resisting, this will be this spring.

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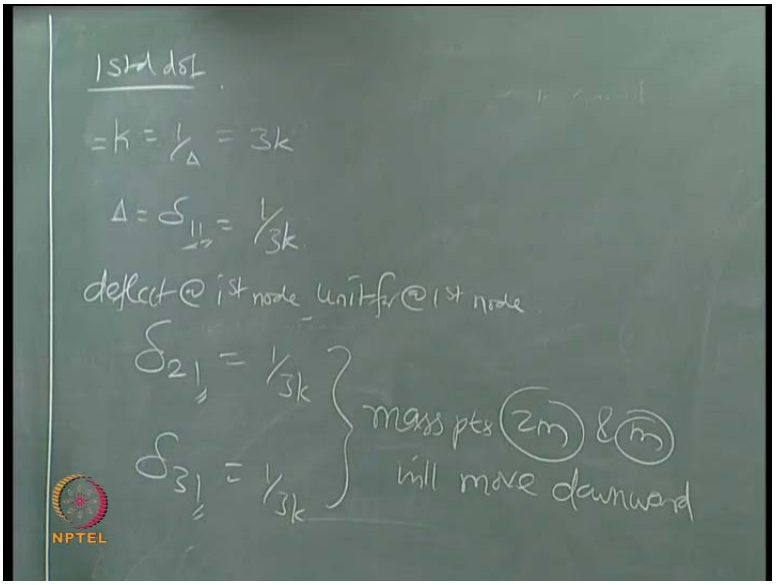

So, we already know stiffness is 1 by displacement of flexibility, which is given in this case as 3k, first degree of freedom, 3k is what is active here. Now, I am interested to find out this as unit force for displacement. So, this is going to be delta 11, which will be 1 by 3k. So, this is nothing but deflection at first node by giving unit force at first node keeping all other nodes constant. I am not changing them.

Now, these two mass, 2m and m, will move solid downwards. So, delta 21 and delta 31 displacement in the second node because of unit force given in the first node. Displacement in the third node because of unit force given in the first node, we derive it column wise, will be all same as 1 by 3k. The reason being the mass points 2m and m will move downward because of this force, just move downwards. Let us do the same operation for the second degree, that is, give unit force at the second coordinate here at x 2 and see what happens.

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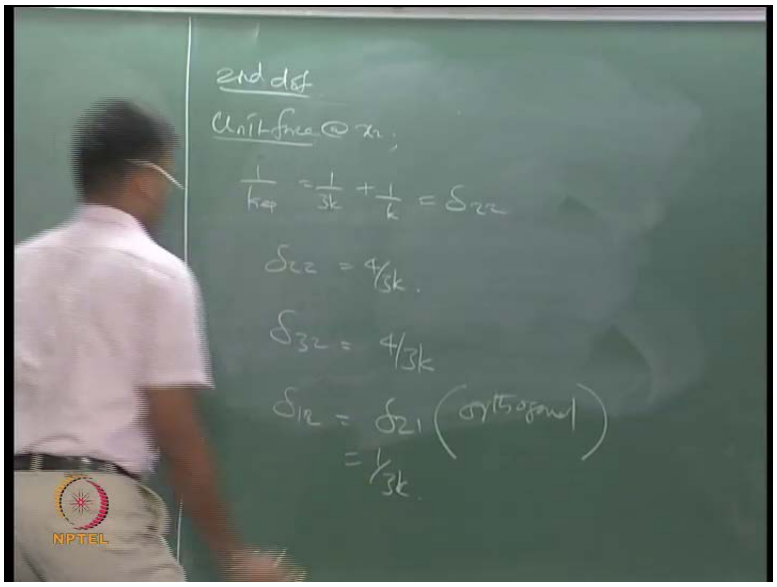

So, I write (( )) unit force at x 2, unit force, remember this, the force is unity, not the displacement. So, as I said, this method is following the advance application of Dunkerley. When I give a unit force here, this method assumes, that these two springs become in series. So, I work out k equal, which is now delta 22. So, delta 22 is 4 by 3k, 1 by stiffness is delta actually, so directly taking it from here. Can you tell me what will be delta 32? The third node is same and what will be delta 12? Sorry, delta 12, delta 12 will be as same as delta 21 because this method assumes, the modes are orthogonal, which is

1 by 3. Let us do for the third degree. I think, you understand why I am writing this because as per the reciprocal theorem, when the modes are supposed to be orthogonal, then this is valid, force caused by the node one or node two is as same as the displacement caused on node 2 by the force node 1. Exactly, what we are doing here is this.

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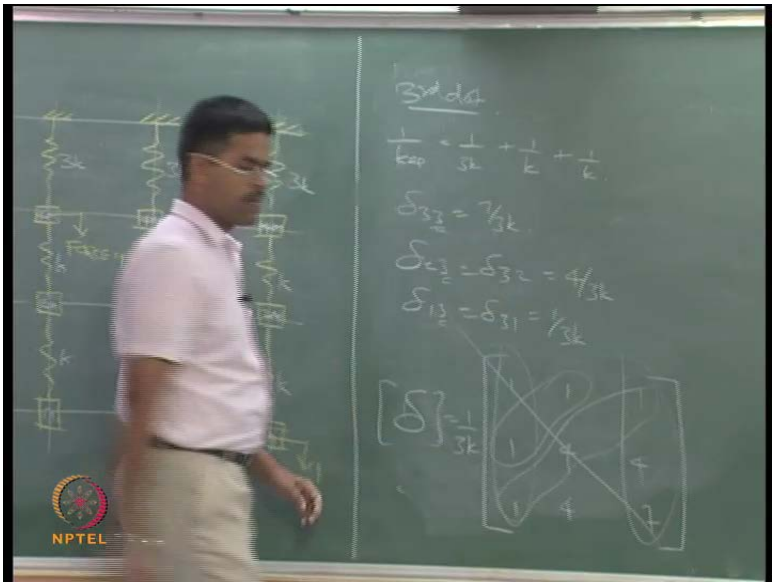

So, third degree, I give unit force here. All the springs will be now in series, can work out k equal 1 by 3k plus 1 by k plus 1 by k. So, delta 33 will be 7 by 3k and delta 23 is same as 32, delta 13 is same as 31. So, 32 means, it is 4 by 3k and 31 means 1 by 3k. So, I have a delta matrix now, which I call influence coefficient matrix, which will be 1 by 3 k of… So, it is symmetric diagonally dominant square, therefore inverse exist. The inverse of this is nothing but stiffness matrix and it is derived column wise.

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For example, look at here, the derived column. Let us quickly check for our interest what is the Dunkerley's frequency for this. What is the Dunkerley's frequency for this? It is nothing but sum of i is equal to 1 to n. In my case, it is 3 m i delta ii only. I must take only the diagonal values and m's are here, 4m, 2m and m, so 1 by omega square m by 3k of 4, 8, 7. Is that right? Is that ok? 4m and its value is 1, therefore 4, this is 2 m, this value is 4 8, this value is 7, this value is 1, so it is 7.

So, what is omega? So many radius per second, k by m of course, sum, is it 0.45 or what? 0.397, this value, 0.397, right. So, this is 0.397 provided m is in kg and k is in Newton per meter. So, this is an extension of Dunkerley, but I want the mode shapes as well and I want the frequencies not from the Dunkerley's, but from the influence coefficient techniques. So, let us now develop the algorithm for finding out omega and phi paralelly using this technique. So, I will remove this.

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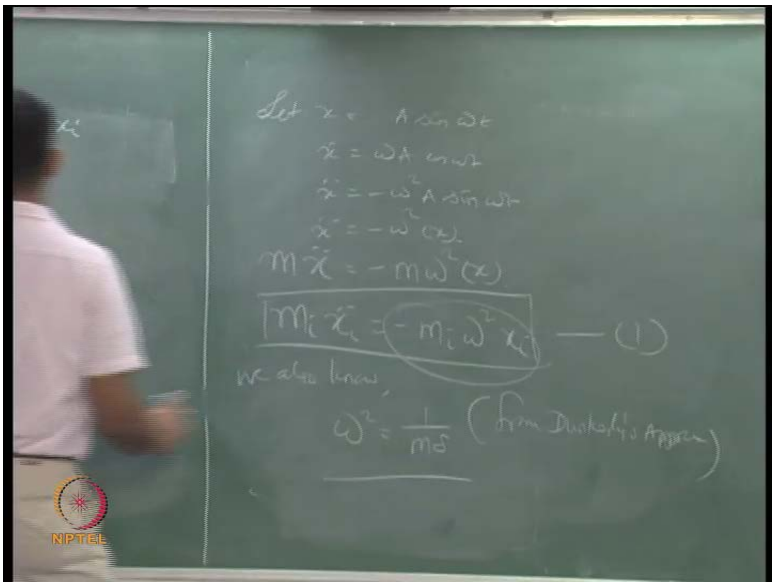

I said in iterative scheme, now let us set an algorithm for this scheme. The displacement d, let us say, A sine omega t x dot cos omega t x double dot sine omega t, where I can say, minus omega square x is x double dot. Let me do not now. The (( )) right, I multiply this with mass on both sides because I am getting the inertia force, what I am applying here the unit force, I want that to be in inertia force. So, I should say, it is m x double dot is minus m omega square of x. So, if I say, for a specific node or specific mode, we also know omega square is given by m a delta i. Is it not? From Dunkerley's approximation this is valid, so substitute back here.

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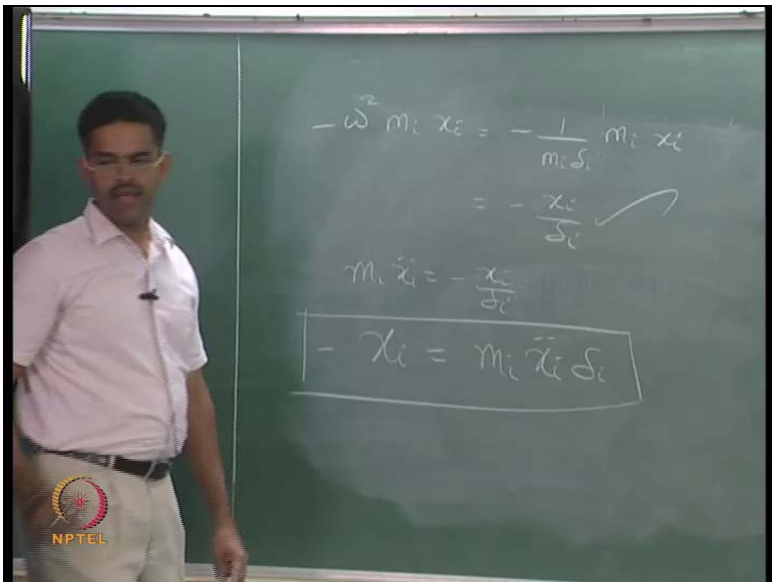

So, so minus omega square m i x i can be written as minus 1 by m i delta i of m i x. So, it is nothing but minus x i, but the psi is, so nothing but m i x i double dot is x i… This is what I am using here, this is what I had here, I am just equating. So, what does it mean is, the control equation is x i is m i x i double dot delta. Now, I want to generate the equation for this for three degree freedom system model for this example, let us see how do we do it. It is a very interesting way of writing, generally people get confused once we write this control algorithm. it is very important, see how we are writing this here. I am sure hundred percent, not, you will not memorize this. If you try to memorize this, you will definitely make a mistake and you will land up in a different answer. There is a very easy thumb rule, short cut to remember this, how to write this equation because I am writing the control equation for iteration.

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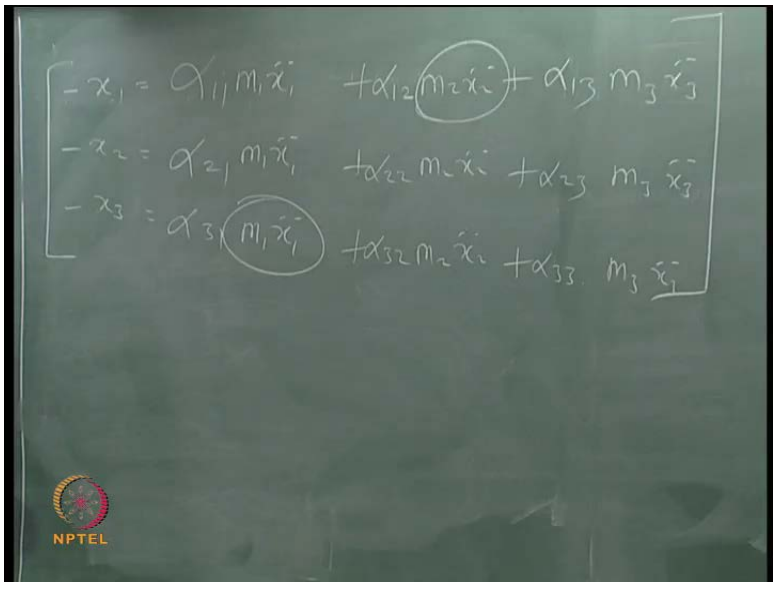

There is an i here, I will expand this. How do I do it is minus x 1, minus x 2, minus x 3, please do not write, just see, minus x 1, minus x 2, minus x 3, I am expanding i. So, instead of delta I am comfortable with alpha, alpha 11, 21, 31. I am writing the first column. Then alpha… Once I have done it, say m 1 x 1 double dot, this is how we will write the equation. For example, if you start writing it like this, one by one, you will tend to make a mistake here, which is not correct or you will tend to make a mistake here thinking, that this is 2, this is 2, which is also not right. So, be very careful in writing this equation. The equation is written in the simple form like do this, write down the

coefficients, then put m 1, m 1, m 1, m 2, m 2, m 2, m 3, m 3, m 3, these are all inertia forces in specific degrees of freedom. So, if you write, this becomes simple.

Now, the question is, where iteration sets in this algorithm? This is my algorithm, where iteration sets here? I will remove this. So, once you know how to write the control equation, which are generated from this principle, because I do not have omegas here, I have only x, I have alphas, which are nothing but deltas, I have these values, I have of course, m 1, m 2 in my problem, but I do not know omega here, but influence coefficient technique should give me omega and phi both. Phi may be the relative displacements of x 1, x 2, x 3, which are nothing but the mode shapes. If I am able to get x 1, x 2, x 3 vector for a specific omega, this is nothing but the mode shapes only. But there is no omega here. What I will do here is, x i double dot, nothing but omega square m i. I will substitute that here, minus will go away, I get a control algorithm now in omega, as well as, x 1, x 2, x 3, iterate it, get this in the next class.

I will show you that in the next class how it is done. I will solve this problem, then we will find omega and phi from this and compare to Dunkerley. You see, I am getting 4.4 phi, whereas Dunkerley gave me (( )) k by… Very close and this can be easily programmed. This method can be easily programmed for n degrees of freedom and this method has no approximation. It is completely, purely analytical procedure, it has no approximation, whereas Dunkerley had an approximation, there is no approximation. The only implicit approximation is, that the mode shapes what you generate now will be orthogonal, that is the condition what I have used, which I have used further also in the next algorithm and solve. Any doubts here? So, we will close it here.